The Graph Minor Algorithm with Parity Conditions

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Abstract— We generalize the seminal Graph Minor algorithm of Robertson and Seymour to the parity version. We give polynomial time algorithms for the following problems:

1) the parity H-minor (Odd K_k-minor) containment problem, and
2) the disjoint paths problem with k terminals and the parity condition for each path,

as well as several other related problems.

We present an $O(m\alpha(m,n)n)$ time algorithm for these problems for any fixed k, where n, m are the number of vertices and the number of edges, respectively, and the function $\alpha(m,n)$ is the inverse of the Ackermann function (see Tarjan [69]).

Note that the first problem includes the problem of testing whether or not a given graph contains k disjoint odd cycles (which was recently solved in [24], [34]), if we fix H to be equal to the graph of k disjoint triangles. The algorithm for the second problem generalizes the Robertson Seymour algorithm for the k-disjoint paths problem.

As with the Robertson-Seymour algorithm for the k-disjoint paths problem for any fixed k, in each iteration, we would like to either use the presence of a huge clique minor, or alternatively exploit the structure of graphs in which we cannot find such a minor. Here, however, we must maintain the parity of the paths and can only use an "odd clique minor". This requires new techniques to describe the structure of the graph when we cannot find such a minor.

We emphasize that our proof for the correctness of the above algorithms does not depend on the full power of the Graph Minor structure theorem [56]. Although the original Graph Minor algorithm of Robertson and Seymour does depend on it and our proof does have similarities to their arguments, we can avoid the structure theorem by building on the shorter proof for the correctness of the graph minor algorithm in [35]. Consequently, we are able to avoid the much of the heavy machinery of the Graph Minor structure theory. Utilizing some results of [35] and [62], [63], our proof is less than 50 pages.

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1. INTRODUCTION

1.1. Graph Minors Algorithm and Parity condition

One of the deepest and most important bodies of work in graph theory is the Graph Minor Theory developed by Robertson and Seymour. The main algorithmic result of the Graph Minor Theory is a polynomial-time algorithm for testing the existence of a fixed minor [55] which, combined with the proof of Wagner's Conjecture [57], implies the existence of a polynomial-time algorithm for deciding any minor-closed graph property. The existence of such a polynomial time algorithm has in turn been used to show the existence of polynomial-time algorithms for several graph problems, some of which were not previously known to be decidable [14]. It also leads to the framework of parameterized complexity developed by Downey and Fellows [13], which is perhaps one of the most active areas in the study of algorithms.

In this paper, we find an efficient algorithm for the parity k-disjoint paths problem. The *parity k-disjoint paths problem* accepts in input a graph G and k triples (s_i, t_i, p_i), $1 \leq i \leq k$, where s_i and t_i are vertices of G, $p_i \in \{1, 0\}$, and $\{s_i, t_i\} \cap \{s_j, t_j\} = \emptyset$. The problem is to determine whether there exist disjoint paths $P_1, \dots, P_k$ such that the ends of P_i are s_i and t_i and the length of P_i modulo 2 is equal to p_i.

Theorem 1.1 *There exists an $O(nm\alpha(m,n))$-time algorithm for the parity k-disjoint paths problem where*

n is the number of vertices, m is the number of edges, and α is the inverse of the Ackermann function.

We actually prove a the existence of an efficient algorithm for a generalization of the parity k-disjoint paths problem. An immediate consequence of this result is efficient algorithms for the following problems (k is a fixed integer).

1) The k disjoint odd cycles problem (i.e., finding k disjoint odd cycles).
2) The parity minor (odd minor) containment problem (i.e., find a minor with parity conditions, see below).
3) The parity H-subdivision problem (i.e., finding an H-subdivision with parity conditions, see below).

For more detailed descriptions of the problems, we refer the reader to the next section.

The k disjoint odd cycles problem was solved in [34]. Recently, Geelen, Gerards, Huynh and Whittle have announced a completed project to extend much of the theory of Graph Minors to a model of group labelled graphs; they are currently writing up the project. Their progress on the disjoint paths problem is reported in Huynh's thesis [24], and they have announced an $O(n^6)$ bound on the runtime for the algorithm for the group labeled k-disjoint paths problem. This implies polynomial time algorithms for the problems above. The proof heavily depends on the Graph Minor structure theorem and its generalization obtained by Geelen, Gerards and Whittle in their research program on matroid minors. As a consequence, their proof is lengthy and highly technical.

1.2. Odd cycles and disjoint paths

Our motivation is to combine and generalize two important problems: the k disjoint odd cycles problem and the k disjoint paths problem.

We begin with a look at the k disjoint odd cycles problem. Finding a minimum vertex cover (or vertex transversal) for the set of odd cycles in a given graph G (which we call the *Odd Cycle Cover Problem*) is a basic problem in both combinatorial optimization and theoretical computer science. Determining a minimum edge cover for the set of odd cycles is equivalent to the *maximum cut problem*. On the positive side, the maximum cut problem is solvable for planar graphs (by Hadlock [23]). Reed et al. [49] presented an $O(mn)$ algorithm to determine whether or not a given graph with m edges and n vertices has an odd cycle cover of order at most k for any fixed k (recently, the time complexity is improved to almost linear time by the first two authors [33]). The k disjoint odd cycles problem may be viewed as the "dual" problem of the odd cycle cover problem (and hence it is also connected to the maximum cut problem). This problem is known to be NP-hard, even for planar graphs, if k is part of input, see [15]. For more details in this context, we refer the reader to the book by Schrijver [64]. It also has direct applications in combinatorial biology (see for example, [50]). We remark that packing *disjoint cycles*, i.e, finding disjoint cycles as many as possible, has been also studied extensively (see [3], [43]). Efficiently finding k disjoint cycles for fixed k is easy. It follows from the grid theorem [51] together with dynamic programming for graphs of bounded tree-width [2], [5]. As remarked before, the k disjoint odd cycles problem was solved recently in [34]. However, the proof requires many of the Graph Minors tools including the structure theorem. Indeed, the proof hinges upon generalizing several of the Graph Minors results to parity versions.

We now turn our attention to the k disjoint paths problem which is a central problem in algorithmic graph theory both in it's vertex and edge disjoint versions. See the surveys [17], [64] as well as the work of Chekuri et al. [6], [7] and of Tardos and Kleiberg [40], [39]).

We now quickly look at previous results on the k disjoint paths problem. If k is as a part of the input of the problem, then this is one of Karp's original NP-complete problems [38], and it remains NP-complete even if G is restricted to be planar (Lynch [44]). The seminal work of Robertson and Seymour says that there exists a polynomial time algorithm (the actual runtime of the algorithm is $O(|V(G)|^3)$. The time complexity is improved to $O(|V(G)|^2)$ in [37] (Reed also gave an unpublished proof, see [47]).

As we see, the problem of finding k disjoint cycles becomes much more difficult when we impose parity conditions on the cycles. Our motivation here is to extend the k disjoint paths problem to the parity version. Let us observe that the k parity disjoint paths problem clearly generalizes the k disjoint paths problem (i.e, without the parity condition) because we can test every possible parity conditions for each path (there are 2^k possibilities), which would give rise to the solution for the k disjoint paths problem. Partial progress has previously been made on the k parity disjoint paths problem. Building on the methods in [31], Kawarabayashi and Reed [32] gave a polynomial time algorithm for the half-integral parity disjoint paths problem where each vertex is allowed to be in at most two of the desired k paths.

1.3. Odd minors

For basic terminology in this paper, we refer the reader to Appendix. Recall that a graph H is a *minor* of G if H can be obtained by contracting and deleting edges in G. Equivalently, H is a minor of G precisely if there are $|V(H)|$ vertex-disjoint trees in G, one tree T_v for each vertex v of H, such that for every edge vw in H there is an edge $e(v,w)$ in G connecting the two corresponding trees T_v and T_w. The graph H is an *odd minor* of G if, in addition, all the vertices of the trees can be two-colored in such a way that (1) the edges within each tree T_v in G are bichromatic, while (2) the edge $e(v,w)$ connecting trees T_v and T_w in G corresponding to each edge vw of H is monochromatic. In particular, the class of odd-H-minor-free graphs (excluding a fixed graph H as an odd minor) is more general than the class of H-minor-free graphs (excluding a fixed graph H as a minor).

Indeed, the class of odd-H-minor-free graphs is strictly more general: the complete bipartite graph $K_{n/2,n/2}$ certainly contains a K_k-minor for $k \leq n/2$, but on the other hand, it does not contain K_k as an odd minor for $k \geq 3$. In fact, any K_k-minor-free graph G is $O(k\sqrt{\log k})$-degenerate, i.e, every induced subgraph has a vertex of degree at most $O(k\sqrt{\log k})$; see [41], [42], [71], [72]. Thus, any K_k-minor-free graph G has $O(k\sqrt{\log k}n)$ edges. On the other hand, the example shows that odd-K_k-minor-free graphs may have $\Theta(n^2)$ edges.

Odd minors play an important role in the field of discrete optimization. A long-standing area of interest in this field is finding conditions under which a given polyhedron has integer vertices, so that integer optimization problems can be solved as linear programs. In the case of a particular set-covering formulation of the maximum-cut problem, there is a structural characterization based on excluding odd minors due to Guenin. See [21], [22], [65]. In general, odd-minor-free graphs have been considered extensively in the graph theory literature (see, e.g., [18], [19], [21], [22], [27], [29], [65], [66]), and there are many recent references in theoretical computer science as well[9], [24], [34], [30], [70].

2. Our main results

We prove that for any fixed positive integer k, there are polynomial time algorithms for the following problems.

The k Parity Disjoint (Rooted) Paths (or k-PDRP)

Input: A graph G, two sets of vertices $S = \{s_1, ..., s_k\}$ and $T = \{t_1, ..., t_k\}$ with $S \cap T = \emptyset$ and values $j_1, ..., j_k$ with $j_i \in \{0, 1\}$.

Output: Are there k vertex disjoint paths $P_1, ..., P_k$ from S to T in G such that P_i has endpoints s_i and t_i and the parity of $|E(P_i)|$ is j_i?

The k Parity (Rooted) Routing (or k-PRR)

Input: A graph G, and a set X of $2k$ vertices.

Output: Solve the k-PDRP for every instance $(G, S, T, \{j_1, ..., j_k\})$ such that $S \cup T = X$.

An easy construction allows us to generalize the algorithm for the k-PDRP to the problem where we allow the paths to share endpoints but insist that they are otherwise disjoint (thus S and T may be multi-sets and may intersect). One can simply make multiple copies of any vertex which appears more than once in $S \cup T$, and then solve the corresponding k-PDRP.

We can use these algorithms as subroutines to solve a number of related problems, whose statements require some definitions.

To *subdivide* an edge e in a graph H, we replace it by a path of length two through a new vertex. A *subdivision* of a graph H is a graph obtained from H by repeatedly subdividing edges. That is, a graph in which each edge of H has been replaced by a path with the same endpoints such that these paths can share endpoints but are otherwise disjoint. We say G contains a *subdivision* of H if there is a subgraph of G isomorphic to a subdivision of H. We refer to the vertices of this subdivision which correspond to the vertices of H as the *branch vertices* of the subdivision.

To test if G has a subdivision of H, where the path corresponding to each edge has some given parity, we only need to test, for each of the $O(n^{|V(H)|})$ injections of $V(H)$ into $V(G)$, whether G has such an subdivision of H where the given injection specifies the branch vertices corresponding to each vertex of H. To do this, however, we need only solve (for $k = |E(H)|$) the extension of the k-PDRP which allows non disjoint multisets for S and T, and which we have just remarked is no more difficult than k-PDRP. Thus, we also have a polynomial time algorithm for the following problem.

The Parity H-Subdivision

Input: A graph G, for each edge e of H, an integer j_e in $\{0, 1\}$.

Output: Is there an subdivision of H in G such that for each edge e of H, the path corresponding to e has parity j_e?

Consider a function im with domain $E(H) \cup V(H)$ such that:

1) $\forall v \in V(H),\ im(v)$ is a tree in G,
2) $\forall e = uv \in E(H),\ im(e)$ is an edge of G with one endpoint in $im(u)$ and the other in $im(v)$,
3) for all distinct v,$v' \in V(H)$, $im(v)$ and $im(v')$ are disjoint,
4) by the last two statements, $\forall e, e' \in E(H), v \in V(H)\ :\ \ im(e)$ is not an edge of $im(v)$ and $im(e) \neq im(e')$.

We refer to a function im satisfying the above conditions as a *model* of H in G. Note that G has H as a minor if and only if it contains a model of H.

By the *image* of a vertex v we mean the tree $im(v)$. By the *image* of an edge e we mean the edge $im(e)$. For any cycle C of H there is a unique cycle contained in the union of the images of $E(C)$ and $V(C)$. We call this cycle the *image* of C.

For any fixed graph H, we also give a polynomial time algorithm which solves the following decision problem (again, an $O(m\alpha(m,n)n)$ time algorithm):

The Parity H-Minor Containment

Input: A graph G, for each cycle C of H, an integer j_C in $\{0, 1\}$.

Output: Is there a model of H in G such that for each cycle C of H, the image of C has parity j_C?

Remarks. One case of particular interest here is that in which j_C is the parity of C. Thus G has an odd minor of H if it contains a model of H, where the image of each cycle has the same parity as the cycle. This is equivalent to requiring that we can two color the vertices of G so that the edges within the vertex images are bichromatic but the images of the edges are monochromatic. Thus our algorithm implies that we can test odd K_k-minor containment. We also note that if we take H as k disjoint triangles, then this problem is equivalent to finding k disjoint odd cycles. Thus our algorithm implies that we can test the k disjoint odd cycles problem.

The results are summarized in the following theorem, generalizing Theorem 1.1.

Theorem 2.1 *For a fixed integer k, there is an $O(m\alpha(m,n)n)$ time algorithm for the k-PRDP, the k-PRR and the parity H-minor containment problem for H with $|V(H)| = k$ (thus the odd K_k-minor containment problem and the k disjoint odd cycles problem), where n, m are the number of vertices and the number of edges, respectively, and the function $\alpha(m,n)$ is the inverse of the Ackermann function (see by Tarjan [69]).*

Also, there is an $O(m\alpha(m,n)n^{k+1})$ algorithm for the parity H-subdivision problem.

In the proof of Theorem 2.1, we actually resolve "the parity-folio relative to the terminal set S", which includes all the problems k-PRDP, k-PRR and the parity H-minor containment problem (thus the odd K_k-minor containment problem and the k disjoint odd cycles problem). This concept generalizes the idea of a "folio" introduced by Robertson and Seymour in their Graph Minor algorithm [55]. Thus, our results extend all the concepts and main algorithmic results of [55] to the analogous parity versions.

2.1. Our contribution

Theorem 2.1 improves the run-time for all the problems above. Recall that Huynh's thesis [24] gives a polynomial time algorithm for the k disjoint paths problem in group labeled graphs. They are primarily concerned with obtaining a polynomial bound, and so at several steps maintain more general structures than are strictly necessary. They claim that the run-time can be bounded by $O(n^6)$. This translates immediately to an $O(n^6)$-time algorithm for the k-parity disjoint paths problem. Looking at the other problems, the result would imply an $O(n^{k+6})$-time algorithm for the k disjoint odd cycles problem, and $O(n^{|H|+6})$-time algorithm for parity H-minor containment and parity H-minor subdivision. Our algorithm improves the runtime and in addition gives the first fixed parameter tractable algorithm for parity H-minor containment as well. Perhaps, more importantly, our proof is much simpler and avoids many of the technical issues that arise in previous approaches to these problems. We explain this in more detail below.

Our proof differs substantially from previous work on these problems. The original proof of correctness of the Robertson Seymour algorithm for the disjoint paths problem requires the full power of the Graph Minor Theory. More precisely, it is not terribly hard to reduce the problem to the case when the input graph has no large clique minor. However, the analysis of this case requires the development of the structure theorem for graphs with no clique minor. This structure theorem [56, Theorem 1.3] describes the structure of graphs excluding a fixed graph as a minor and lies at the heart of the Robertson Seymour theory of minors. At a high level, the theorem says that every such a graph can be decomposed into a collection of graphs each of which can "almost" be embedded into a bounded-genus surface, combined in a tree structure. Much of the Graph Minors series of articles is devoted to the proof of this theorem (recently, a much shorter proof of this seminal structure theorem is given in [36]).

Previous work on the parity problems above proceeded by generalizing the Robertson Seymour techniques to their corresponding parity versions. Instead, we are able to prove our results without using many of the technicalities in the original proof of Robertson and Seymour. Specifically, we avoid the use of the structure theorem by building on the shorter proof for the correctness of the graph minor algorithm in [35]. Utilizing some results in [35], our proof is less than 50 pages.

3. How the algorithm works

In this subsection, we sketch our algorithm. For simplicities sake, we describe the algorithm in terms of the k-PRDP, and we omit here the generalization to parity folios.

At a very high level, we follow the approach of Robertson and Seymour for the regular k-disjoint paths problem. Robertson-Seymour's algorithm is based on the following two cases: either the input graph G has bounded tree-width (bounded by some function of k) or else it has large tree-width.

Case 1. Tree-width of G is bounded.

In this case, one can apply a dynamic programming argument to a tree-decomposition of bounded tree-width, see [2], [5], [55].

Case 2. Tree-width of G is large.

This second case again breaks into two cases depending on whether G has a large clique minor or not.

Case 2.1. G has a large clique minor.

If there exist disjoint paths from the terminals to this clique minor, then we can use this clique minor to link up the terminals in any desired way. Otherwise, there is a small cut set such that the large clique minor is cut off from the terminals by this cut set. In this case, we can prove that there is a vertex v in the clique minor which is *irrelevant*, i.e., the given k disjoint paths problem is feasible in G if and only if it is also feasible in $G - v$. We then recursively apply the algorithm to $G - v$.

Case 2.2. G does not have a huge clique minor.

In this case, one can prove that, after deleting a bounded number of vertices, there is a huge subgraph which is essentially planar. Moreover, this huge planar subgraph itself has very large tree-width. This makes it possible to prove that the "middle" vertex v of this planar subgraph is irrelevant. Again, we recursively apply the algorithm to $G - v$.

The analysis of Cases 1 and 2.1 is relatively easy. It is the analysis of Case 2.2 that gives rise to the majority of the work. The analysis of this case requires the whole series of Graph Minor papers and the structure theorem of [56].

Let us now come back to our algorithm for the parity disjoint paths problem. Let G be a graph and $\{s_1, \dots, s_k\}$, $\{t_1, \dots, t_k\}$, p_i, $1 \leq i \leq k$ be given. If the graph G has bounded tree-width (bounded by some function of k), then we can again apply a dynamic programming approach to determine whether there exist disjoint paths forming a solution to the problem. Thus we may assume that Case 2 happens.

As with the Robertson-Seymour algorithm to solve the k disjoint paths problem for any fixed k, as in Case 2 above, we would like to either use a huge clique minor, or exploit the structure of graphs in which we cannot find such a minor. Here, however, we must maintain the parity of the paths and can only use an "odd clique minor". We must also describe the structure of those graphs in which we cannot find such a minor and discuss how to exploit it. To do so, we need to consider the following three cases for the above case 2.

- (i) The graph contains a huge clique minor and a smaller, but still big odd clique minor.
- (ii) The graph contains a huge clique minor, but no big odd clique minor.
- (iii) The graph contains a wall of huge height, but no huge clique minor.

Let us first discuss the case (i). Robertson and Seymour also excluded a huge clique minor for the k disjoint paths problem, and that step is not hard. To maintain path parities, we first need to exclude a big odd clique minor. We will prove that if there is a big odd clique minor, then either there are desired paths, or there is an irrelevant vertex (as in Case 2.1) in the odd clique minor. We present this argument in more detail in the next subsection.

One question is: how do we find a big odd clique minor? It turns out that we have a nice structure theorem which tells us that if we have a huge clique minor, then either we can get a big odd clique minor or else we can get a vertex set X of bounded size (depending on k) such that the component of $G - X$ containing most of the nodes of the huge clique minor is "essentially" bipartite. This is proved below in Theorems 3.2 with help of the recent result by Geelen et al. [19]. We discuss these issues more thoroughly in the next subsection.

3.1. Using a huge clique minor

We first see the relationship between huge clique minors and large even clique minors. Geelen et al. [19] proved the following result.

Theorem 3.1 *Suppose G has an even clique model of order at least $16k$. Then either G has an odd clique model of order k or G has a vertex set X with $|X| < 8k$ such that the block F intersecting at least $8k$ disjoint nodes (that consist of disjoint trees) of the even clique model in $G - X$ is bipartite.*

The next result follows from Geelen et al. [19].

Theorem 3.2 *Suppose G has a $K_{32k\sqrt{\log k}}$-model. Then G has an even K_{16k}-model. Consequently, either G has an odd clique model of order k or G has a vertex set X with $|X| < 8k$ such that the block F intersecting at least $8k$ disjoint nodes (that consist of disjoint trees) of the even clique model in $G-X$ is bipartite. Moreover, these $8k$ nodes are also nodes of the clique model.*

For the sake of brevity, we omit here the proof of Theorem 3.2.

We now see how the existence of a large odd clique model which is not separated by a small cut from a set X of vertices allows us to solve every possible routing on X. Recall that given a subset $S \subseteq V(G)$, an *S-cut* is a pair (A, B) of non-empty subsets A, B of $V(G)$ such that $V(G) = A \cup B$, $S \subseteq A$, $B - A \neq \emptyset$, and G has no edge joining $A - B$ to $B - A$. The order of the S-cut is $|A \cap B|$. The proof of the next lemma is inspired by that of Robertson and Seymour [55], and we believe this lemma itself is of independent interest. This proof was also motivated by the proofs in [25], [26].

Lemma 3.3 *Let G be a graph and $S = \{s_1, \ldots, s_k, t_1, \ldots, t_k\}$ be a set of $2k$ vertices. Suppose G has an odd-K_{7k}-model and no S-cut of order less than $2k$ such that $B - A$ contains at least one node of the odd K_{7k}-model. Then G has k vertex disjoint paths $P_1, \ldots, P_k$ such that P_i joins s_i and t_i for $1 \leq i \leq k$, and we can specify any parity (i.e, even or odd) for P_i.*

Proof: We will prove the following slightly stronger statement, which immediately implies Lemma 3.3:

> (*) Let G be a graph and $S = \{s_1, \ldots, s_k, t_1, \ldots, t_k\}$ be a set of $2k$ vertices. Suppose G contains $7k$ vertex disjoint non-empty subgraphs D_i for $1 \leq i \leq 7k$ such that each D_i is either connected or each of its components meets S. Moreover each D_i is adjacent to all D_j ($i \neq j$) which do not meet S. Suppose G has no S-cut (A, B) of order less than $2k$ with at least one D_i in $B - A$. Then G has k vertex disjoint paths $P_1, \ldots, P_k$ such that P_i joins s_i and t_i for $1 \leq i \leq k$, and each P_i goes through at least three D_i's that do not intersect any P_j for $i \neq j$ nor S.

We note that any three nodes of the odd clique model contains an odd cycle. Thus, if we prove the statement (*), then assuming that each D_i is a node of a given odd clique model of order $7k$, we can control the parity of each path P_i via the odd cycle in three of D_i's that do not intersect any P_j for $i \neq j$ nor S. Therefore, the statement (*) implies Lemma 3.3.

We prove the statement (*) by induction on $|V(G)|$. It is easy to check that the statement (*) is true for $|V(G)| = 7k$. Let G be a minimal counterexample to (*), that is, take G such that $|V(G)| + |E(G)|$ is as small as possible. If all D_i's contain no edges except for $E(S)$, then by Hall's theorem, there is a perfect matching between S and $G - S$, and the result easily follows since $|G - S| \geq 5k$ and $G - S$ consists of a clique.

Thus it remains to show that there is no D_i that contains an edge in $E(G) - E(S)$. For suppose $e \in E(G) - E(S)$ is such an edge contained in some D_i. If we contract e, then the resulting graph is either no longer a counterexample or has a S-cut of order exactly $2k - 1$. In the former case, we are done. So, we may assume that there exists an S-cut (A, B) in G of order exactly $2k$ with the endpoints of e contained in $A \cap B$. Since each D_i is adjacent to all D_j's ($i \neq j$) which do not meet S, D_i is adjacent to at least $5k$ of D_j's. Hence for any i, D_i cannot be contained in $A - B$. Let $S' = A \cap B$,

$G' = G[B]$, and let $D'_i = D_i \cap G'$ for $1 \leq i \leq 7k$. Note that $S \subseteq A$. If $S' = S$, then $G - e$ would form a smaller counterexample. So $A - B \neq \emptyset$. By the assumption of the statement (*) and Menger's theorem, there exist $2k$ disjoint paths from S to S'. Thus, G', S' and D'_i for $1 \leq i \leq 7k$ satisfy the assumption of (*) and therefore G' satisfies the conclusion of (*) by our choice of a minimal counterexample. This would imply that G satisfies the conclusion as well, a contradiction. We conclude that there is no such an edge e.

This completes the proof of (*). ■

Lemma 3.3 allows us to find a an irrelevant vertex given a sufficiently large odd clique minor. The next theorem actually shows something slightly stronger.

Theorem 3.4 *Let G be a graph and $S = \{s_1, \ldots, s_k\}$, $T = \{t_1, \ldots, t_k\}$, and $j_1, \ldots, j_k$ form an instance of the k-PDRP. Suppose G has an odd-K_{7k}-model. Then either G has a solution to the problem instance, or else we can get a minimal $S \cup T$-cut (A, B) of order at most $2k - 1$ with at least one node of the odd-K_{7k}-model in $B - A$. If we replace B by K_{4k} such that this K_{4k} contains all the vertices in $A \cap B$, then the resulting graph has a solution to the problem instance if and only if the original graph does.*

An immediate consequence is that given a model of an odd K_{7k+1} in an instance of the k-PDRP, we can find an irrelevant vertex. Thus we turn our attention to graphs which have a large clique minor, but no large odd minor. The next theorem allows us to handle the case when the graph contains a huge clique minor but no large odd clique minor.

Theorem 3.5 *Let G be a graph and let $X \subseteq V(G)$ with $|X| = 2k$. If G has a K_l-model, where $l \geq 2^{60k} \times 60k$, but does not have an odd clique model of order $7k+1$, then there is a vertex v in G satisfying the following. For any partition of X into sets S and T, $S = \{s_1, \ldots, s_k\}$, $T = \{t_1, \ldots, t_k\}$, and choice of values $j_1, \ldots, j_k$ form an instance of the k-PDRP, the vertex v is irrelevant to the problem instance. Moreover, such a vertex v can be found in $g(l)m$ time for some function g of l.*

3.2. When there is no big clique minor

We now come to the point where we are given an instance of the k-PDRP which does not contain a huge clique minor, as in (iii) above. By a result of Robertson and Seymour [55], after deleting bounded number of vertices, there is a huge almost planar graph W of large treewidth. In this case, we follow the approach as in the above Case 2.2 of the Robertson Seymour algorithm for disjoint paths. Let us emphasis again that our proof in this case does not depend on the full power of the Graph Minor structure theorem [56]. We reduce the problem of finding an irrelevant vertex in the graph to the following theorem.

Theorem 3.6 *Let G be a graph and let $P_1, \ldots, P_k$ be disjoint paths in G such that $\bigcup_1^k V(P_i) = V(G)$. There exists a value $w = w(k)$ such that if $tw(G) \geq w$, then there exists a vertex v such that $G - v$ has disjoint paths $P'_1, \ldots, P'_k$ such that for all $1 \leq i \leq k$, P_i and P'_i have the same endpoints and furthermore, the length of P_i and P'_i are the same modulo 2.*

Thus, the algorithmic problem of finding an irrelevant vertex reduces to the theoretical problem of showing the existence of a an irrelevant vertex to a given instance of the parity disjoint paths problem.

Utilizing some results in [35], we are able to avoid the much of the heavy machinery of the Graph Minor structure theory and reduce the proof of Theorem 3.6 to the case when the graph G is embedded in a surface of bounded genus (bounded by a function of k). The remainder of the argument hinges upon showing Theorem 3.6 in the case that the graph is embedded in a fixed surface.

We assume for the rest of this discussion that the graph G is embedded in a surface of genus bounded by a function of k. We extend a result for graphs on a surface in [52] to the parity version. These results may be of independent interest. An *odd face* of the embedding is a facial cycle of odd length. The analysis is split into cases, depending on whether there exist many odd faces which are pairwise "far apart" in the embedded graph. Alternatively, there exist a bounded number of discs of bounded size which cover all the odd faces in the embedding of G.

In the first case, we use a deep result by Schrijver [62], [63] to show that we can find a desired solution via odd faces. In the second case, we delete each of the discs covering all the odd faces. Deleting each disc will break our paths up into a larger number of paths, but since each disc has bounded size, the number of new paths can grow only by a bounded number. After deleting every disc, the remaining graph will be embedded in a surface with a bounded number of cuffs. Thus, we see we may restrict our attention to the following situation.

> G is embedded into a surface of Euler genus g with bounded number of cuffs, say l

cuffs, such that there are no odd faces. G contains disjoint paths $P'_1, \ldots, P'_{k'}$ for some k' bounded by a function of k such that $\bigcup_1^{k'} V(P'_i) = V(G)$.

Although there are no odd faces in G, G may not be bipartite because some noncontractible cycle may have odd length. However, it is well-known that two paths (with the same endpoints) of the same homotopic type of the surface with Euler genus g and l cuffs have the same parity. Thus a path with a specific homotopic type of the surface determines the parity of the path, and hence we do not have to worry about the parity of the path but we just care for the homotopic type of the path. This allows us to use the above mentioned result by Schrijver [62], [63], with which we can control the homotopic type of a path.

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APPENDIX

We now recall definitions for a K_p-model, an even K_p-model and an odd K_p-model. A graph G contains a K_p-*model* if there exists a function σ with domain $V(K_p) \cup E(K_p)$ such that

1) for each vertex $v \in V(K_p)$, $\sigma(v)$ is a connected subgraph of G, and the subgraphs $\sigma(v)$ ($v \in V(K_p)$) are pairwise vertex-disjoint, and
2) for each edge $e = uv \in E(K_p)$, $\sigma(e)$ is an edge $f \in E(G)$, such that f is incident in G with a vertex in $\sigma(u)$ and with a vertex in $\sigma(v)$.

Thus G contains a K_p-minor if and only if G contains a K_p-model. The order of a complete graph model is the order of the complete graph. We call the subgraph $\sigma(v)$ ($v \in V(K_p)$) a *node* of the K_p-model.

We say that a K_p-model is *even* if the union of the nodes of the K_p-model forms of a bipartite graph. We also say that a K_p-model is *odd* if for each cycle C in the union of the nodes of the K_p-model, the number of edges in C that belong to nodes of the K_p-model is even. Equivalently, the model is odd if the nodes can be properly two colored so that the edges $\sigma(uv)$ are monochromatic for all edges uv in K_p.

A *separation* of a graph G is a pair of subgraphs (A, B) of G such that $G = A \cup B$ and $E(A \cap B) = \emptyset$. The *order* of the separation (A, B) is $|V(A) \cap V(B)|$.

Tree-width and Brambles.: A *bramble* β is a set of trees every two of which intersect or are joined by an edge (thus a clique minor model is a bramble whose elements are disjoint). The *order* of a bramble β, denoted $ord(\beta)$, is the minimum size of a hitting set of its elements (that is, a set H of vertices intersecting the vertex set of each tree of β). Clearly every clique model (minor) of order l is a bramble of order l. Also for any set W of vertices, the set β_W of trees of G containing more than half the vertices of W is a bramble since any two such trees intersect. We now characterize graphs which have no brambles of order l, using tree decompositions.

A tree decomposition of a graph G consists of a tree T and a subtree S_v of T for each vertex v of G such that if uv is an edge of G then S_u and S_v intersect. For each node t of the tree, we let W_t be the set of vertices v of G such that $t \in S_v$. We let H_t be the graph obtained from the subgraph of G induced by W_t by adding an edge between x and y if there is some s such that $x, y \in W_s \cap W_t$. The width of a tree decomposition is the maximum of $|W_t|$ over the nodes t of T.

It is not hard to see that for every bramble β and every tree decomposition there is a node t such that W_t is a hitting set for β. This implies that the tree width of G is at least the maximum order of a bramble. Seymour, and Thomas [68] showed that this bound is tight, proving:

Theorem A.1 *The maximum order of a bramble in G is equal to its tree width.*

One of the most important result concerning the tree-width is the fact that large tree width guarantees existence of grid-minor or a wall.

Grid minors and Walls.: Let us recall that an r-*wall* is a graph which is isomorphic to a subdivision of the graph W_r with vertex set $V(W_r) = \{(i,j) \mid 1 \le i \le r, 1 \le j \le r\}$ in which two vertices (i,j) and (i',j') are adjacent if and only if one of the following possibilities holds:

(1) $i' = i$ and $j' \in \{j-1, j+1\}$.
(2) $j' = j$ and $i' = i + (-1)^{i+j}$.

We can also define an $(a \times b)$-wall in a natural way. It is easy to see that if G has an $(a/2 \times b)$-wall, then it has an $(a \times b)$-grid minor, and conversely, if G has an $(a \times b)$-grid minor, then it has an $(a/2 \times b)$-wall.

Let us recall that the $(a \times b)$-grid is the Cartesian product of paths $P_a \Box P_b$.

One of the most important results concerning the tree-width is the main result in [51] which says the following.

Theorem A.2 *For any r, there exists a constant $f(r)$ such that if G has tree-width at least $f(r)$ (equivalently a bramble B of order at least $f(r)$), then G contains an r-wall W.*

The best known upper bound for $f(r)$ is given in [12], [47], [61]. It is 20^{5r^5}. The best known lower bound is $\Theta(r^2 \log r)$, see [61].